S14 C4 UK

1. A curve C has the equation

$$x^3 + 2xy - x - y^3 - 20 = 0$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(5)

(b) Find an equation of the tangent to C at the point $(3, -2)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

(2)

$$\frac{d}{dx}(x^3+2xy-x-y^3-20)=0$$

$$3x^2 + 2x\frac{dy}{dx} + 2y - 1 - 3y^2\frac{dy}{dx} = 0$$

$$3x^2+2y-1 = (3y^2-2x)\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2+2y-1}{3y^2-2x}$$

$$(3,-2) \quad M_t = \frac{22}{6} = \frac{11}{3}$$

$$y+2 = \frac{11}{3}(x-3) \Rightarrow 3y+6 = 11x-33$$

$$11x-3y-39=0$$

2. Given that the binomial expansion of $(1 + kx)^{-4}$, $|kx| < 1$, is

$$1 - 6x + Ax^2 + \ldots$$

(a) find the value of the constant k,

(2)

(b) find the value of the constant A, giving your answer in its simplest form.

(3)

$$(1+kx)^{-4} = 1 + (-4)(kx) + \frac{(-4)(-5)}{2}(kx)^2$$

$$= 1 - 4kx + 10k^2x^2$$

$$\therefore \ \underline{\underline{k = 1.5}} \qquad A = 10k^2 = \underline{\underline{22.5}}$$

3.

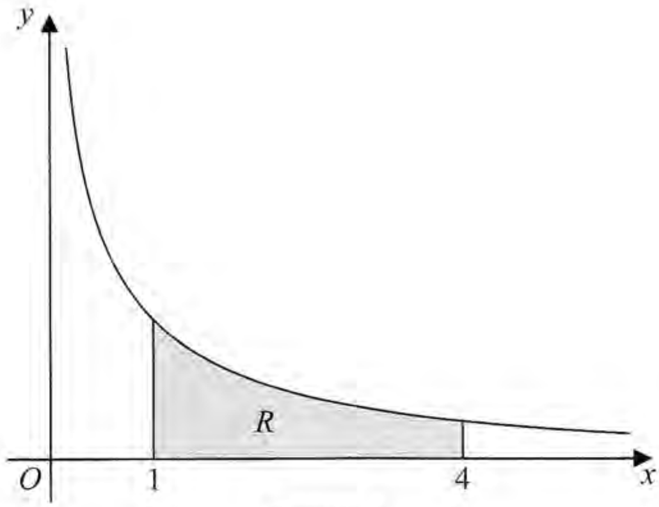


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{10}{2x + 5\sqrt{x}}$, $x > 0$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis, and the lines with equations $x = 1$ and $x = 4$

The table below shows corresponding values of x and y for $y = \dfrac{10}{2x + 5\sqrt{x}}$

$h=1$

x	1	2	3	4
y	1.42857	0.90326	0.68212	0.55556

(a) Complete the table above by giving the missing value of y to 5 decimal places.

(1)

(b) Use the trapezium rule, with all the values of y in the completed table, to find an estimate for the area of R, giving your answer to 4 decimal places.

(3)

(c) By reference to the curve in Figure 1, state, giving a reason, whether your estimate in part (b) is an overestimate or an underestimate for the area of R.

(1)

(d) Use the substitution $u = \sqrt{x}$, or otherwise, to find the exact value of

$$\int_1^4 \frac{10}{2x + 5\sqrt{x}}\, dx$$

(6)

b) $\frac{1}{2}(1)\left[1.42857 + 2(0.90326 + 0.68212) + 0.55556\right)$

$= 2.5774$ b) over estimate, curve will lie under trapezia.

2

$u = \sqrt{x} \quad u = x^{\frac{1}{2}}$

$u^2 = x$

$x = 1 \quad u = 1$

$x = 4 \quad u = 2$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$2x^{\frac{1}{2}}\,du = dx$$

$$2u\,du = dx$$

$$\int_1^2 \frac{10}{2u^2+5u} \times 2u\,du$$

$$= \int_1^2 \frac{20}{2u+5}\,du = 10\int_1^2 \frac{2}{2u+5}\,du$$

$$= 10\left[\ln|2u+5|\right]_1^2$$

$$= 10\left[\ln 9 - \ln 7\right] = 10\ln\frac{9}{7}$$

4.

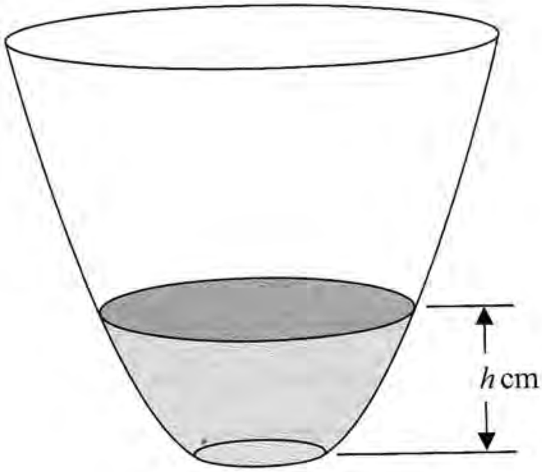


Figure 2

A vase with a circular cross-section is shown in Figure 2. Water is flowing into the vase.

When the depth of the water is h cm, the volume of water V cm^3 is given by

$$V = 4\pi h(h+4), \quad 0 \leqslant h \leqslant 25$$

Water flows into the vase at a constant rate of 80π cm^{3}s^{-1}

Find the rate of change of the depth of the water, in cm s^{-1}, when $h = 6$

(5)

$V = 4\pi h^2 + 16\pi h \qquad \frac{dV}{dt} = 80\pi$

$\frac{dV}{dh} = 8\pi h + 16\pi$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{8\pi h + 16\pi} \times 80\pi$

$\frac{dh}{dt} = \frac{80\pi}{8\pi(h+2)} = \frac{10\pi}{\pi(h+2)}$

$h = 6 \qquad \frac{dh}{dt} = \frac{10}{8\pi}$

$= 1.25$

5.

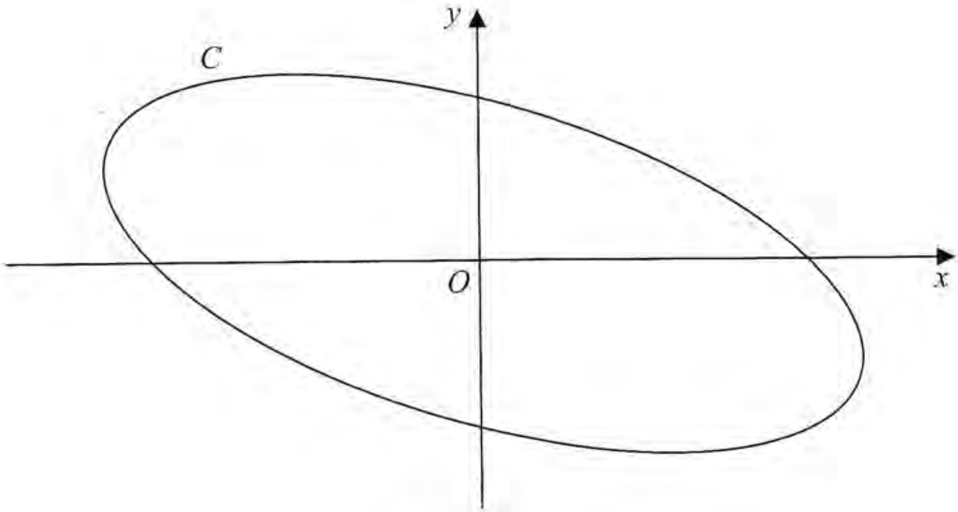


Figure 3

Figure 3 shows a sketch of the curve C with parametric equations

$$x = 4\cos\left(t + \frac{\pi}{6}\right), \quad y = 2\sin t, \quad 0 \leqslant t < 2\pi$$

(a) Show that

$$x + y = 2\sqrt{3}\cos t$$

(3)

(b) Show that a cartesian equation of C is

$$(x + y)^2 + ay^2 = b$$

where a and b are integers to be determined.

(2)

a) $x = 4\cos t\cos\frac{\pi}{6} - 4\sin t\sin\frac{\pi}{6}$

$x = 2\sqrt{3}\cos t - 2\sin t$

$\therefore x + y = 2\sqrt{3}\cos t - 2\sin t + 2\sin t \quad \therefore x + y = 2\sqrt{3}\cos t$

$(x+y)^2 = (2\sqrt{3}\cos t)^2 = 12\cos^2 t$

$y^2 = 4\sin^2 t \Rightarrow 4 - y^2 = 4 - 4\sin^2 t$

$\Rightarrow 4 - y^2 = 4\cos^2 t$

$\Rightarrow 12 - 3y^2 = 12\cos^2 t$

$\therefore (x+y)^2 = 12 - 3y^2 \quad \therefore (x+y)^2 + 3y^2 = 12$

2

6. (i) Find

$$\int x e^{4x} \, dx$$

(3)

(ii) Find

$$\int \frac{8}{(2x-1)^3} \, dx, \quad x > \frac{1}{2}$$

(2)

(iii) Given that $y = \frac{\pi}{6}$ at $x = 0$, solve the differential equation

$$\frac{dy}{dx} = e^x \operatorname{cosec} 2y \operatorname{cosec} y$$

(7)

i) $u = x \qquad v = \frac{1}{4}e^{4x} \qquad = \frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x} dx$

$u' = 1 \qquad v' = e^{4x}$

$= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$

ii) $\int 8(2x-1)^{-3} dx$

$= -2(2x-1)^{-2} + C$

$u = (2x-1)^{-2} \quad \times -2$

$\frac{du}{dx} = -2(2x-1)^{-3} \times 2$

$= -4(2x-1)^{-3} \quad \times -2$

iii) $\frac{dy}{dx} = e^x \operatorname{cosec} 2y \operatorname{cosec} y$

$\int \frac{1}{\operatorname{cosec} 2y \operatorname{cosec} y} dy = \int e^x dx$

$\int \sin 2y \sin y \, dy = e^x + C$

$\int 2\sin^2 y \cos y \, dy = e^x + C$

$u = (\sin y)^3 \times \frac{2}{3}$

$\frac{du}{dx} = 3\sin^2 y \cos y \times \frac{2}{3}$

$\frac{2}{3}\sin^3 y = e^x + C$

$x = 0 \; y = \frac{\pi}{6} \quad \frac{2}{3}\left(\frac{1}{2}\right)^3 = 1 + C \quad \therefore C = \frac{-11}{12}$

$\frac{2}{3}\sin^3 y = e^x - \frac{11}{12}$

7.

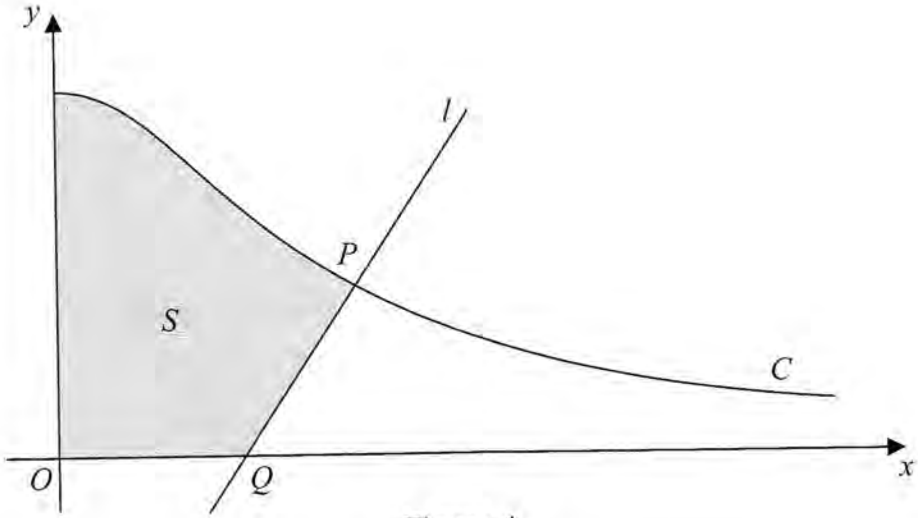


Figure 4

Figure 4 shows a sketch of part of the curve C with parametric equations

$$x = 3\tan\theta, \quad y = 4\cos^2\theta, \qquad 0 \leqslant \theta < \frac{\pi}{2}$$

The point P lies on C and has coordinates (3, 2).

The line l is the normal to C at P. The normal cuts the x-axis at the point Q.

(a) Find the x coordinate of the point Q.

(6)

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the x-axis, the y-axis and the line l. This shaded region is rotated 2π radians about the x-axis to form a solid of revolution.

(b) Find the exact value of the volume of the solid of revolution, giving your answer in the form $p\pi + q\pi^2$, where p and q are rational numbers to be determined.

[*You may use the formula* $V = \frac{1}{3}\pi r^2 h$ *for the volume of a cone.*]

(9)

$y = 4\cos^2\theta$ $\quad x = 3\tan\theta$ $\quad \frac{dy}{dx} = \frac{8\sin\theta\cos\theta}{3\sec^2\theta}$

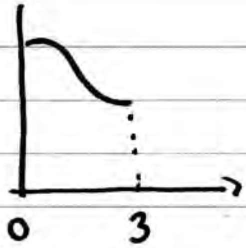
$\frac{dy}{d\theta} = -8\sin\theta\cos\theta$ $\quad \frac{dx}{d\theta} = 3\sec^2\theta$ $\quad \frac{dy}{dx} = \frac{-8\sin\theta\cos^3\theta}{3}$

at P $x = 3$ $\quad 3 = 3\tan\theta$

$\tan\theta = 1$ $\theta = \frac{\pi}{4}$ $\therefore M_t = \frac{-8\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^3}{3} = \frac{8 \times 4}{3 \times 16} = \frac{-2}{3}$

$m_n = +\frac{3}{2}$ $\quad y-2=\frac{3}{2}(x-3)$ $\quad y=0 \Rightarrow x-3=\frac{-4}{3} \therefore x=\frac{5}{3}$

b)



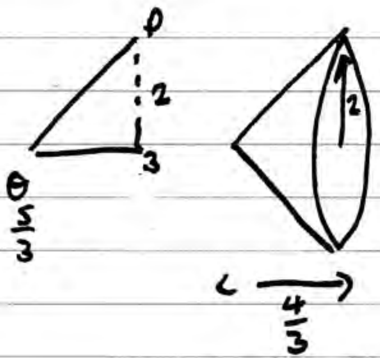
$$V=\pi\int_{0=x}^{3=x} y^2 \frac{dx}{d\theta}\, d\theta \qquad \begin{matrix} x=3 & \theta=\frac{\pi}{4} \\ x=0 & \theta=0 \end{matrix}$$

$$V=\pi\int_{0=x}^{3=x} 16\cos^4\theta \times 3\sec^2\theta\, d\theta$$

$$V=\pi\, 48\int_{0=x}^{3=x}\cos^2\theta\, d\theta = 48\pi\int_0^{\frac{\pi}{4}} \frac{1}{2}+\frac{1}{2}\cos 2\theta\, d\theta$$

$$V=24\pi\int_0^{\pi/4} 1+\cos 2\theta\, d\theta$$

$$V=24\pi\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = 24\pi\left[\left(\frac{\pi}{4}+\frac{1}{2}\right)-(0)\right] = 6\pi^2+12\pi$$



$$V=\frac{1}{3}\pi(2)^2\left(\frac{4}{3}\right)=\frac{16}{9}\pi$$

$$\therefore S = 62\pi^2+12\pi-\frac{16}{9}\pi$$

$$S=62\pi^2+\frac{92}{9}\pi$$

8. Relative to a fixed origin O, the point A has position vector $\begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$

and the point B has position vector $\begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}$

The line l_1 passes through the points A and B.

(a) Find the vector $\overrightarrow{AB}$.

(2)

(b) Hence find a vector equation for the line l_1

(1)

The point P has position vector $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

Given that angle PBA is θ,

(c) show that $\cos\theta = \dfrac{1}{3}$

(3)

The line l_2 passes through the point P and is parallel to the line l_1

(d) Find a vector equation for the line l_2

(2)

The points C and D both lie on the line l_2

Given that $AB = PC = DP$ and the x coordinate of C is positive,

(e) find the coordinates of C and the coordinates of D.

(3)

(f) find the exact area of the trapezium $ABCD$, giving your answer as a simplified surd.

(4)

a) $a = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$ $b = \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix}$ $\vec{AB} = b - a = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

b) $l_1 = \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

c) $p = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ $\vec{BP} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}$ $|\vec{BP}| = \sqrt{1^2+1^2+5^2}$
$= \sqrt{27} = 3\sqrt{3}$

$|\vec{BA}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

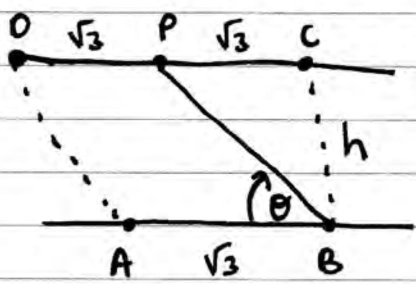
$\vec{BA} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$\vec{BP} \cdot \vec{BA} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix} = -1 - 1 + 5 = 3$

$\therefore \cos\theta = \left(\frac{\vec{BP} \cdot \vec{PA}}{|\vec{BP}||\vec{BA}|} \right) = \frac{3}{(3\sqrt{3})(\sqrt{3})} = \frac{3}{9} = \frac{1}{3}$ #

d) $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

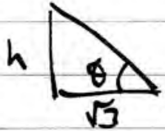
e)



$\therefore C = p + \vec{AB} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$

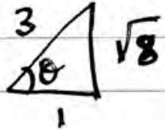
$D = p - \vec{AB} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

f) Area $= (\sqrt{3} + 2\sqrt{3}) \times \frac{1}{2}h$



$h = \sqrt{3}\tan\theta$

$\cos\theta = \frac{1}{3}$



$\therefore \tan\theta = \sqrt{8}$ $\quad \therefore h = \sqrt{8}\sqrt{3}$

$\therefore$ Area $= 3\sqrt{3} \times \frac{\sqrt{8}}{2}\sqrt{3}$

$= \frac{9}{\cancel{2}} \times \cancel{2}\sqrt{2} = \underline{9\sqrt{2}}$